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Energy quantization for biharmonic maps

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Abstract. In the present work we establish an energy quantization (or energy identity) result for solutions to scaling invariant variational problems in dimension 4 which includes biharmonic maps (extrinsic and intrinsic). To that end we first establish an angular energy quantization for solutions to critical linear 4th order elliptic systems with antisymmetric potentials. The method is inspired by the one introduced by the authors previously in "Angular energy quantization for linear elliptic systems with antisymmetric potentials and applications" (2011) for 2nd order problems.

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1 Introduction

Let N be a C^3 closed submanifold of $\mathbb{R}^k$ (i.e. N is compact without boundary). Let B_1 the unit ball of $\mathbb{R}^n$ and $u \in W^{1,2}(B_1, N)$. Then we can define the Dirichlet energy of u as

$$D(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 \, dx.$$

The critical points of D are the so-called harmonic maps for which an extensive theory has been developed. In particular, when $n = 2$ since in that case the functional is conformally invariant, it has been proved that the harmonic maps have some special properties, in particular an energy quantization for sequences of bounded energy, see [13] for instance.

In this paper, we consider still quadratic scaling invariant problems but in dimension $n = 4$ this time. In that case, there are several ways to define an equivalent of the Dirichlet functional. Since we look for a scaling invariant quadratic functional, the gradient has to be replaced by some expression involving second derivatives. The simplest example is given by

$$E(u) = \frac{1}{4}\int_{B_1} |\Delta u|^2 \, dx.$$

The critical point of this functional are called *extrinsic biharmonic maps*. The term extrinsic comes from the fact that this functional (and consequently its critical points) depends on the choice of the embedding of N into $\mathbb{R}^k$. Trying to remedy to this lack of intrinsic nature of the problem, one can instead consider the following functional:

$$I(u) = \frac{1}{4}\int_{B_1} |(\Delta u)^T|^2\, dx,$$

where $(\Delta u)^T$ is the projection of Δu onto $T_u N$ (indeed $(\Delta u)^T := \sum_k D_{\partial_{x_k}} \partial_{x_k} u$ where D is the pull back by u of the Levi-Civita connection ∇ on N for the induced metric). The critical point of I will be called *intrinsic biharmonic maps*. One can further introduce other functionals sharing similar properties and we refer to [12] for more examples. The Euler Lagrange equations satisfied by the biharmonic maps have been computed in particular in [17]. One shows that $u \in W^{2,2}(B_1, N)$ is an extrinsic (resp. intrinsic) biharmonic map if and only if u satisfies

$$T_e(u) \equiv \Delta^2 u - \Delta(B(u)(\nabla u, \nabla u)) - 2\nabla \cdot \langle \Delta u, \nabla P(u)\rangle + \langle \Delta(P(u)), \Delta u\rangle = 0,$$

respectively

$$\begin{aligned} T_i(u) \equiv\ & \Delta^2 u - \Delta(B(u)(\nabla u, \nabla u)) - 2\nabla \cdot \langle \Delta u, \nabla P(u)\rangle + \langle \Delta(P(u)), \Delta u\rangle \\ & - P(u)(B(u)(\nabla u, \nabla u)\nabla_u B(u)(\nabla u, \nabla u)) \\ & - 2B(u)(\nabla u, \nabla u) B(u)(\nabla u, \nabla P(u)) = 0, \end{aligned}$$

where P and B are the orthogonal projection onto $T_u N$ and the second fundamental form of N.[1] Since our result applies indistinctly to extrinsic as well as to intrinsic biharmonic maps, except when it is necessary, in what follow *we will indifferently employ the denomination biharmonic map for both extrinsic biharmonic map and intrinsic biharmonic map*. We observe that these equations are of the form,

$$\Delta^2 u = \sum_{\substack{\alpha_1+\cdots+\alpha_4=4 \\ 0\leq \alpha_i<4}} c_\alpha(u)\, \partial^{\alpha_1} u\, \partial^{\alpha_2} u\, \partial^{\alpha_3} u\, \partial^{\alpha_4} u,$$

which make them critical in dimension 4 for $W^{2,2}$ in the sense that classical L^p-theory can be directly applied to this equation for proving regularity or compactness results assuming u is in $W^{2,p}(B_1)$ with $p > 2$ but such an approach fails in $W^{2,2}$. The critical nature of an elliptic problem is characterized by possible loss of compactness at isolated points. In order to fully describe this concentration-compactness phenomenon one has to understand "how much" energy is lost at

[1] See section 2 for precise definitions.

these isolated points. *Energy quantization* means that the energy lost corresponds exactly to the sum of the energies of the so called *bubbles* – or rescaled elementary solutions on S^4 – concentrating at these points. The word *quantization* refers to the fact that the bubbles cannot have arbitrary small energy and in some problems it is even known that they can realize only a discrete set of values.

Our main result in this paper is the energy quantization result for biharmonic maps. In fact we are proving something stronger considering more generally sequences of approximate solutions of biharmonic maps. To that end we need the following definition.

Definition 1.1. Let N be a C^3-submanifold of $\mathbb{R}^k$, $p \geq 1$, $f \in L^p(B_1, \mathbb{R}^k)$ and $u \in W^{2,2}(B_1, N)$. The map u is f-approximate biharmonic if u satisfies

$$T_i(u) = f \quad \text{or} \quad T_e(u) = f.$$

The reason why we need N at least C^3 is made clear in Section 2 when we rewrite the equation in term of orthogonal projections onto $T_u N$. Hence, we are in a position to state our main result.

Theorem 1.2. *Let N be a C^3-submanifold of $\mathbb{R}^k$, $p > 1$, $f_n \in L^p(B_1, \mathbb{R}^k)$ and let $u_n \in W^{2,2}(B_1, N)$ be a sequence of f_n-approximate biharmonic maps with bounded energy, i.e.*

$$\int_{B_1} \left(|\nabla^2 u_n|^2 + |\nabla u_n|^4 + |f_n|^p \right) dz \leq M. \tag{1.1}$$

Then there exists $f \in L^p(B_1, \mathbb{R}^k)$, $u_\infty \in W^{2,1}(B_1, N)$ an f-approximate biharmonic map and

(i) *$\omega^1, \dots, \omega^l$ some biharmonic maps of $\mathbb{R}^4$ to N,*

(ii) *$a_n^1, \dots, a_n^l$ a family of converging sequences of points of B_1,*

(iii) *$\lambda_n^1, \dots, \lambda_n^l$ a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$u_n \to u_\infty \quad \text{in } W^{2,q}_{\text{loc}}(B_1 \setminus \{a_\infty^1, \dots, a_\infty^l\})$$

for all $q < \frac{2p}{2-p}$ if $p < 2$, for any q otherwise, and

$$\left\| \nabla^2 \left(u_n - u_\infty - \sum_{i=1}^{l} \omega_n^i \right) \right\|_{L^2_{\text{loc}}(B_1)} + \left\| \nabla \left(u_n - u_\infty - \sum_{i=1}^{l} \omega_n^i \right) \right\|_{L^4_{\text{loc}}(B_1)} \to 0,$$

where $\omega_n^i = \omega^i(a_n^i + \lambda_n^i\, .)$. *Moreover, if* N *is* C^{l+3} *and the map* f_n *is bounded in* $C^{l,\eta}(B_1, \mathbb{R}^k)$*, then the convergence of* u_n *to* u_∞ *is in* $C^{l+4,\nu}(B_1 \setminus \{a_\infty^1, \dots, a_\infty^l\})$ *for any* $0 \le \nu < \eta$.

Observe that for a sequence of biharmonic maps into a smooth manifold the convergence holds in C^∞_{loc}. Such a result was already known for intrinsic biharmonic maps, see [6] and [7], or for extrinsic biharmonic maps into a sphere, see [19]. Here, the method employed seems particularly robust since it can be applied equally for both extrinsic and intrinsic biharmonic maps but it applies moreover to a larger class of scaling invariant problems. As an illustration of this fact we prove that the method applies to the following general lagrangians:

$$\int_{B_1} \left(|\Delta u|^2\, dx + u^*\Omega\right) \quad \text{or} \quad \int_{B_1} \left(|(\Delta u)^T|^2\, dx + u^*\Omega\right), \tag{1.2}$$

where Ω is an arbitrary smooth 4-form of $\mathbb{R}^k$.

The method we use goes first through the proof of an angular energy quantization result[2] for sequences of solutions to the general critical 4th order elliptic system with antisymmetric potentials introduced by Lamm and Rivière [10]. We follow in fact the approach that we originally introduced in [11] for second order problems. We have good reasons to think that the method could further be extended for proving a general energy quantization result for polyharmonic maps in critical dimension (see the ε-regularity for polyharmonic maps in [4] and [3] for the general case, see also [14]).

As an immediate consequence of Theorem 1.2, we get the asymptotic behavior of biharmonic maps flow. A weak solution to the extrinsic biharmonic map flow is a map $u \in W^{2,2}([0, +\infty[\times B_1, N)$ satisfying

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta^2 u = \Delta(B(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla P(u)\rangle & \\ \qquad\qquad - \langle \Delta(P(u)), \Delta u\rangle & \text{on } [0, +\infty[\times B_1, \\ u = u_0 & \text{on } \{0\} \times B_1, \end{cases} \tag{1.3}$$

where $u_0 \in W^{2,2}(B_1, N)$. Several existence results have been established for (1.3), see for instance [9] for small initial data or [2] and [18] for solution with finitely many singular times and arbitrary initial data. All these solutions satisfy the energy identity

$$2\int_0^T \int_{B_1} \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt + \int_{B_1} |\Delta u|^2\, dx \le \int_{B_1} |\Delta u_0|^2\, dx \quad \text{for all } T \ge 0. \tag{1.4}$$

[2] See the end of Section 5 for a precise statement.

Corollary 1.3. *Let N be a C^3-submanifold of $\mathbb{R}^k$ and $u_0 \in W^{2,1}(B_1, N)$ and let $u \in W^{2,2}([0, +\infty[\times B_1, N)$ be a global solution of* (1.3) *satisfying the energy inequality* (1.4)*. Then there exist t_n a sequence of positive real such that $t_n \to +\infty$, a biharmonic map $u_\infty \in W^{2,1}(B_1, N)$, $l \in \mathbb{N}$, $\omega^1, \dots, \omega^l$ some biharmonic maps of $\mathbb{R}^4$ to N and $a_n^1, \dots, a_n^l$ a family of points of B_1 converging to $a_\infty^1, \dots, a_\infty^l$ such that*

$$u(t_n, .) \to u_\infty \quad \text{on } W^{2,p}_{\mathrm{loc}}(B_1 \setminus \{a_\infty^1, \dots, a_\infty^l\}) \quad \text{for all } p \geq 1$$

and

$$\left\| \nabla^2 \left(u(t_n, .) - u_\infty - \sum_{i=1}^{l} \omega_n^i \right) \right\|_{L^2_{\mathrm{loc}}(B_1)} + \left\| \nabla \left(u(t_n, .) - u_\infty - \sum_{i=1}^{l} \omega_n^i \right) \right\|_{L^4_{\mathrm{loc}}(B_1)} \to 0,$$

where $\omega_n^i = \omega^i(a_n^i + \lambda_n^i \,.)$.

In fact, thanks to (1.4), we easily prove that there exists t_n such that $u(t_n, .)$ satisfies the hypothesis of Theorem 1.2 with $p = 2$.

The paper is organized as follows: in Section 2, we rewrite the equations in order to apply the theory of Lamm and Rivière, in Section 3 we recall the main results of Lamm and Rivière and we prove an ε-regularity result for biharmonic maps, in Section 4 we derive the key estimate in Lorentz space for the angular derivatives in a annular region of arbitrary conformal type, finally in Section 5 we prove our main result postponing technical result to Sections 6 and 7.

2 Biharmonic equation in normal form

Let $N \subset \mathbb{R}^k$ be a C^3-submanifold, there exists $\delta > 0$ such that $\Pi : N_\delta \to N$, the nearest point projection map, is well defined and C^3, where

$$N_\delta = \{y \in \mathbb{R}^k \mid d(y, N) \leq \delta\}.$$

Let, for $y \in N$, $P(y) \equiv \nabla\Pi(y) : \mathbb{R}^k \to T_y N$ be the orthogonal projection, and $P^\perp(y) \equiv \mathrm{Id} - \nabla\Pi(y) : \mathbb{R}^k \to (T_y N)^\perp$. In the following, we will write P (resp. $P^\perp$) instead of $P(y)$ (resp. $P^\perp(y)$) and we will identify these linear transformations with their matrix representations in $\mathcal{M}_k$. We also note that these projections are in C^2 and therefore their composition with u, that we keep denoting respectively P and $P^\perp$, are in $W^{2,2}(B_1, \mathcal{M}_k)$ as soon as u is in $W^{2,2}(B_1, N)$.

Finally, let $B(\,.\,)(\,.\,,\,.\,)$ be the second fundamental form of $N \subset \mathbb{R}^k$, which is defined by

$$B(y)(Y, Z) = D_Y P^{\perp}(y)(Z) \quad \text{for all } y \in N,\ Y, Z \in T_y N.$$

We know that, see [16], that $u \in W^{2,2}(B_1, N)$ is an extrinsic biharmonic map if and only if

$$\Delta^2 u \perp T_u N \text{ almost everywhere,}$$

which can be rewritten as follows:

$$\begin{aligned} \Delta^2 u &= P^{\perp}\Delta^2 u \\ &= \operatorname{div}(P^{\perp}\nabla\Delta u) - \nabla P^{\perp}\nabla\Delta u. \end{aligned} \tag{2.1}$$

Then we rewrite the second term of the right hand side as follows:

$$\begin{aligned} \nabla P^{\perp}\nabla\Delta u &= \nabla P^{\perp} P^{\perp}\nabla\Delta u + \nabla P^{\perp} P\nabla\Delta u \\ &= \nabla P^{\perp} P^{\perp}\nabla\Delta u - P^{\perp}\nabla P\nabla\Delta u \\ &= 2\nabla P^{\perp} P^{\perp}\nabla\Delta u + (\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla\Delta u. \end{aligned} \tag{2.2}$$

But

$$\begin{aligned} 2\nabla P^{\perp} P^{\perp}\nabla\Delta u &= 2\nabla P^{\perp} P^{\perp}\nabla\Delta u - 2\nabla P^{\perp}\nabla \operatorname{div}(P^{\perp}\nabla u) \\ &= -2\nabla P^{\perp}\nabla P^{\perp}\Delta u + 2\operatorname{div}(\nabla P^{\perp}(\nabla P^{\perp}\nabla u)) \\ &\qquad - 2\Delta P^{\perp}\nabla P^{\perp}\nabla u. \end{aligned} \tag{2.3}$$

Thanks to (2.1), (2.2) and (2.3), we get

$$\begin{aligned} \Delta^2 u &= \operatorname{div}(P^{\perp}\nabla\Delta u) - \operatorname{div}(2\nabla P^{\perp}(\nabla P^{\perp}\nabla u)) \\ &\qquad + 2\nabla P^{\perp}\nabla P^{\perp}\Delta u + 2\Delta P^{\perp}\nabla P^{\perp}\nabla u \\ &\qquad - (\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla\Delta u \\ &= \Delta(P^{\perp}\Delta u) - \operatorname{div}(\nabla P^{\perp}\Delta u + 2\nabla P^{\perp}(\nabla P^{\perp}\nabla u)) \\ &\qquad + 2\nabla P^{\perp}\nabla P^{\perp}\Delta u + 2\Delta P^{\perp}\nabla P^{\perp}\nabla u \\ &\qquad - (\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla\Delta u, \end{aligned}$$

which finally gives the equation of extrinsic biharmonic maps

$$\begin{aligned} \Delta^2 u &= -\Delta(\nabla P^{\perp}\nabla u) - \operatorname{div}(\nabla P^{\perp}\Delta u) \\ &\qquad + 2\nabla P^{\perp}\nabla(\nabla P^{\perp}\nabla u) + 2\nabla P^{\perp}\nabla P^{\perp}\Delta u \\ &\qquad - (\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla\Delta u. \end{aligned} \tag{2.4}$$

For intrinsic biharmonic maps, we need to add some tangent terms, see [17] for details, which gives

$$\begin{aligned}\Delta^2 u = -\Delta(\nabla P^{\perp}\nabla u) &- \operatorname{div}(\nabla P^{\perp}\Delta u)\\ &+ 2\nabla P^{\perp}\nabla(\nabla P^{\perp}\nabla u) + 2\nabla P^{\perp}\nabla P^{\perp}\Delta u\\ &- (\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla\Delta u\\ &+ P\big(\nabla P^{\perp}\nabla u\nabla(\nabla P^{\perp}\nabla u)\big)\\ &+ 2\nabla P^{\perp}\nabla u\nabla P^{\perp}\nabla P.\end{aligned} \tag{2.5}$$

Proposition 2.1. *Equations* (2.4) *and* (2.5) *can be rewritten in the form*

$$\Delta^2 u = \Delta(V\nabla u) + \operatorname{div}(w\nabla u) + \nabla\omega\nabla u + F\nabla u, \tag{2.6}$$

where

$$\begin{aligned}&V \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), && w \in L^2(B_1, \mathcal{M}_k),\\ &\omega \in L^2(B_1, \mathrm{so}_k), && F \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)\end{aligned}$$

with

$$\begin{aligned}|V| &\le C|\nabla u|,\\ |F| &\le C|\nabla u|\big(|\nabla^2 u| + |\nabla u|^2\big) \ \textit{almost everywhere},\\ |w| + |\omega| &\le C\big(|\nabla^2 u| + |\nabla u|^2\big)\end{aligned} \tag{2.7}$$

where C is a positive constant which depends only on N.

Proof of Proposition 2.1. We give a proof for equation (2.4), the intrinsic case will follow easily.

On the one hand, we proceed to the following Hodge decomposition:

$$dPP^{\perp} - P^{\perp}dP = d\alpha + d^*\beta,$$

where $\alpha \in W^{1,2}(B_1, \mathrm{so}_k)$, $\beta \in W_0^{1,2}(B_1, \Lambda^2(R^4) \otimes \mathcal{M}_k)$. Hence α and β satisfy

$$\Delta\alpha = \Delta P P^{\perp} - P^{\perp}\Delta P \quad \text{and} \quad \Delta\beta = dP \wedge dP^{\perp} - dP^{\perp} \wedge dP.$$

Then $\alpha \in W^{2,2}(B_1, \mathrm{so}_k)$, $d^*\beta \in W_0^{2,(\frac{4}{3},1)}(B_1, \Lambda^2(R^4) \otimes \mathcal{M}_k)$ and we get

$$\begin{aligned}(\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla\Delta u &= d\Delta\alpha\nabla u + \Delta d^*\beta\nabla u + \Delta((\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla u)\\ &\qquad - 2\operatorname{div}(\nabla(\nabla P P^{\perp} - P^{\perp}\nabla P)\nabla u)\\ &= \nabla\omega_1\nabla u + F_1\nabla u + \Delta(V_1\nabla u) + \operatorname{div}(w_1\nabla u),\end{aligned}$$

with $\omega_1 \in L^2(B_1, \mathrm{so}_k)$, $F_1 \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$, $V_1 \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$ and $w_1 \in L^2(B_1, \mathcal{M}_k)$.

On the other hand, we have

$$2\nabla P^{\perp}\nabla(\nabla P^{\perp}\nabla u) = F_2 \nabla u,$$

with

$$F_2^l = 2\frac{\partial P^{\perp}}{\partial y^l}\nabla(\nabla P^{\perp}\nabla u) \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$$

and

$$2\nabla P^{\perp}\nabla P^{\perp}\Delta u = F_3 \nabla u,$$

with

$$F_3^l = 2\frac{\partial P^{\perp}}{\partial y^l}\nabla P^{\perp}\Delta u \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4),$$

which achieves the proof. □

For general Lagrangian of the form (1.2), the equation becomes

$$T_e(u) = H\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}\right) \quad \text{or} \quad T_e(u) = H\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}\right),$$

where H is the 4-form on $\mathbb{R}^k$ into $\mathbb{R}^k$ defined by

$$d\Omega(U, V, W, X, Y) = U_i H^i(V, W, X, Y) \quad \text{for all } U, V, W, X, Y \in \mathbb{R}^k.$$

Hence we have

$$H\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}\right) = F\nabla u,$$

with $F \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$.

3 Preliminaries

First, we recall the main result of [10] that provides a divergence form to elliptic 4th order system of the kind (2.6) under small energy assumption. This will be one of the main tools in order to obtain the estimate needed for the energy quantization.

Theorem 3.1 ([10, Theorem 1.4]). *There exist constants $\varepsilon > 0$ and $C > 0$ depending only on N such that the following holds: Let $V \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$, $w \in L^2(B_1, \mathcal{M}_k)$, $\omega \in L^2(B_1, \mathrm{so}_k)$ and $F \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$ such that*

$$\|V\|_{W^{1,2}} + \|w\|_2 + \|\omega\|_2 + \|F\|_{L^2 \cdot W^{1,2}} < \varepsilon.$$

Then there exist $A \in L^\infty \cap W^{2,2}(B_1, \mathcal{G}l_k)$ *and* $B \in W^{1,\frac{4}{3}}(B_1, \mathcal{M}_k \otimes \Lambda^2\mathbb{R}^4)$ *such that*

$$\nabla\Delta A + \Delta A V - \nabla A w + A(\nabla\omega + F) = \operatorname{curl} B,$$

and

$$\begin{aligned}\|A\|_{W^{2,2}} + d(A, \mathcal{SO}_n) + \|B\|_{W^{1,\frac{4}{3}}} \\ \leq C\big(\|V\|_{W^{1,2}} + \|w\|_2 + \|\omega\|_2 + \|F\|_{L^2\cdot W^{1,2}}\big).\end{aligned}$$

Thanks to the previous theorem, we are in a position to rewrite equations of the form (2.6) in divergence form.

Theorem 3.2 ([10, Theorem 1.2 and 1.4]). *There exist constants* $\varepsilon > 0$ *and* $C > 0$ *depending only on* N *such that if* $u \in W^{2,2}(B_1, \mathbb{R}^k)$ *satisfies*

$$\Delta^2 u = \Delta(V\nabla u) + \operatorname{div}(w\nabla u) + \nabla\omega\nabla u + F\nabla u + f,$$

where

$$\begin{aligned}&V \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), \qquad w \in L^2(B_1, \mathcal{M}_k), \quad \omega \in L^2(B_1, \mathrm{so}_k),\\ &F \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), \quad f \in L^1(B_1, \mathbb{R}^k)\end{aligned}$$

with

$$\|V\|_{W^{1,2}} + \|w\|_2 + \|\omega\|_2 + \|F\|_{L^2\cdot W^{1,2}} < \varepsilon,$$

then there exist $A \in L^\infty \cap W^{2,2}(B_1, \mathcal{G}l_k)$ *and* $B \in W^{1,\frac{4}{3}}(B_1, \mathcal{M}_k \otimes \Lambda^2\mathbb{R}^4)$ *such that*

$$\begin{aligned}\|A\|_{W^{2,2}} + d(A, \mathcal{SO}_n) + \|B\|_{W^{1,\frac{4}{3}}} \\ \leq C\big(\|V\|_{W^{1,2}} + \|w\|_2 + \|\omega\|_2 + \|F\|_{L^2\cdot W^{1,2}}\big)\end{aligned}$$

and

$$\begin{aligned}\Delta(A\Delta u) = \operatorname{div}\big(&2\nabla A\Delta u - \Delta A\nabla u + Aw\nabla u + \nabla A(V\nabla u)\\ &- A\nabla(V\nabla u) - B\nabla u\big) + Af.\end{aligned}$$

A first consequence of the previous theorem is the ε-regularity for biharmonic maps. It can also be compared with the corresponding result established for second order problems in [11, Theorem 3.2].

Theorem 3.3. *Let* $p > 1$. *There exist constants* $\varepsilon > 0$ *and* $C_p > 0$ *such that the following hold:*

(i) (ε-regularity) *If* $u \in W^{2,2}(B_1, \mathbb{R}^k)$, $f \in L^p(B_1, \mathbb{R}^k)$, $V \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$, $w \in L^2(B_1, \mathcal{M}_k)$, $\omega \in L^2(B_1, \mathrm{so}_k)$ *and* $F \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$ *satisfy* (2.7) *and*

$$\|\nabla^2 u\|_2 + \|\nabla u\|_4 \leq \varepsilon,$$

with u a solution of

$$\Delta^2 u = \Delta(V\nabla u) + \operatorname{div}(w\nabla u) + \nabla\omega\nabla u + F\nabla u + f \quad \text{on } B_1,$$

then we have $u \in W^{2,\bar p}(B_{\frac12}, \mathbb{R}^k)$, *where* $\bar p = \frac{2p}{2-p}$ *if* $p < 2$ *else any* $\bar p \geq 2$ *and*

$$\|\nabla^2 u\|_{L^{\bar p}(B_{\frac12})} + \|\nabla u\|_{L^{2\bar p}(B_{\frac12})} \leq C_p\big(\|\nabla^2 u\|_{L^2(B_1)} + \|\nabla u\|_{L^4(B_1)} + \|f\|_p\big).$$

Moreover, if N *is smooth and* $f \in C^{l,\eta}$ *for* $l \in \mathbb{N}$ *and* $\eta > 0$, *then we can replace* $W^{4,\bar p}$ *by* $C^{l+4,\eta}$.

(ii) (Energy gap) *If* $u \in W^{2,2}(\mathbb{R}^4, \mathbb{R}^k)$, $f \in L^p(\mathbb{R}^4, \mathbb{R}^k)$, $V \in W^{1,2}(\mathbb{R}^4, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$, $w \in L^2(\mathbb{R}^4, \mathcal{M}_k)$, $\omega \in L^2(\mathbb{R}^4, \mathrm{so}_k)$ *and* $F \in L^2 \cdot W^{1,2}(\mathbb{R}^4, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$ *satisfy* (2.7) *and*

$$\|\nabla^2 u\|_2 + \|\nabla u\|_4 \leq \varepsilon,$$

with u a solution of

$$\Delta^2 u = \Delta(V\nabla u) + \operatorname{div}(w\nabla u) + \nabla\omega\nabla u + F\nabla u \quad \text{on } \mathbb{R}^4,$$

then u is identically equal to zero.

The proof of Theorem 3.3 could be achieved almost following [10, Lemma 3.1]. We give however an independent proof of this fact that sheds new lights on the problem.

Proof of Theorem 3.3. Let $0 < \varepsilon < 1$ such that, thanks to (2.7), the hypothesis of Theorem 3.2 is satisfied. Then we can rewrite our equation as

$$\Delta(A\Delta u) = \operatorname{div}(K) + Af,$$

where $A \in L^\infty \cap W^{2,2}(B_1, \mathcal{G}l_k)$ and $K \in L^2 \cdot W^{1,2} \subset L^{\frac43,1}$ satisfy

$$\|A\|_{W^{2,2}} + d(A, \mathcal{SO}_n) + \|K\|_{L^{\frac43,1}} \leq C\big(\|\nabla^2 u\|_2 + \|\nabla u\|_4 + \|V\|_{W^{1,2}} + \|w\|_2 + \|\omega\|_2 + \|F\|_{L^2\cdot W^{1,2}}\big)$$

where C is independent of u.

Let $p \in B_{\frac12}$ and $0 < \rho < \frac12$. We decompose $A\Delta u$ on $B_\rho(p)$ as

$$A\Delta u = C + D,$$

where $C \in W_0^{1,2}(B_\rho(p))$ and $D \in W^{1,2}(B_\rho(p))$. Then C satisfies

$$\Delta C = \operatorname{div}(K) + Af \quad \text{on } B_\rho(p)$$

and D satisfies

$$\Delta D = 0 \quad \text{on } B_\rho(p).$$

Thanks to the standard L^p-theory and Sobolev embeddings, we get

$$\begin{aligned}\left(\int_{B_\rho(p)} |C|^2\,dx\right)^{\frac{1}{2}} &\le C\left(\|K\|_{\frac{4}{3}} + \rho^{\frac{4(p-1)}{p}}\|f\|_p\right)\\ &\le C\left(\varepsilon\|\nabla^2 u\|_2 + \frac{\varepsilon}{\rho}\|\nabla u\|_2 + \rho^{\frac{4(p-1)}{p}}\|f\|_p\right),\end{aligned} \tag{3.1}$$

where C is a positive constant in dependent of u.

Using the fact that D is harmonic, we have that

$$\delta \mapsto \frac{1}{(\delta\rho)^4}\int_{B_{\delta\rho}(p)} |D|^2\,dx$$

is an increasing function and hence for all $\delta \in]0,1[$ we deduce,

$$\int_{B_{\delta\rho}(p)} |D|^2\,dx \le \delta^4 \int_{B_\rho(p)} |D|^2\,dx. \tag{3.2}$$

We then decompose the map u as follows: $u = E + F$ where $E \in W_0^{1,4}(B_\rho(p))$ and $F \in W^{1,4}(B_\rho(p))$ satisfy

$$\Delta E = A^{-1}(C + D) \quad \text{on } B_\rho(p)$$

and F satisfies

$$\Delta F = 0 \quad \text{on } B_\rho(p).$$

Thanks to the standard L^p-theory and Sobolev embeddings, we get

$$\frac{1}{\rho}\left(\int_{B_\rho(p)} |\nabla E|^2\,dx\right)^{\frac{1}{2}} \le C\left(\left(\int_{B_\rho(p)} |C|^2\,dx\right)^{\frac{1}{2}} + \left(\int_{B_\rho(p)} |D|^2\,dx\right)^{\frac{1}{2}}\right), \tag{3.3}$$

where C is a positive constant in dependent of u.

The function

$$\delta \mapsto \frac{1}{(\delta\rho)^4}\int_{B_{\delta\rho}(p)} |\nabla F|^2\,dx$$

is increasing since F is harmonic and we have again, for all $\delta \in]0,1[$,

$$\frac{1}{(\delta\rho)^2}\int_{B_{\delta\rho}(p)} |\nabla F|^2\,dx \le \frac{\delta^2}{\rho^2}\int_{B_\rho(p)} |\nabla F|^2\,dx. \tag{3.4}$$

Then, thanks to (3.1), (3.2), (3.3) and (3.4), for δ and ε small enough (with respect to some constant independent of u), we have

$$\int_{B_{\delta\rho}(p)}\left(|\nabla^2 u|^2+\frac{1}{(\delta\rho)^2}|\nabla u|^2\right)dx \\ \leq \frac{1}{2}\int_{B_\rho(p)}\left(|\nabla^2 u|^2+\frac{1}{\rho^2}|\nabla u|^2\right)dx + C\delta^{\frac{4(p-1)}{p}}\|f\|_p^2.$$

Iterating this inequality gives the following Morrey type estimate: there exist $\alpha>0$ and $C>0$ such that

$$\sup_{p\in B_{\frac12},\,0<\rho<\frac12}\rho^{-\alpha}\left(\int_{B_\rho(p)}\left(|\nabla^2 u|^2+\frac{1}{\rho^2}|\nabla u|^2\right)dx\right)\leq C\|f\|_p.$$

Then

$$\sup_{p\in B_{\frac12},\,0<\rho<\frac12}\rho^{-\alpha}\int_{B_\rho(p)}|\Delta^2 u|\,dx\leq C\|f\|_p.$$

Then a classical estimate on Riesz potentials gives, for all $p\in B_{\frac13}$

$$|\Delta u|(p)\leq (C\|f\|_p)\frac{1}{|x|^2}*\chi_{B_{\frac12}}|\Delta^2 u| + C\|\nabla^2 u\|_{L^2(B_1)},$$
$$|\nabla u|(p)\leq (C\|f\|_p)\frac{1}{|x|}*\chi_{B_{\frac12}}|\Delta^2 u| + C\|\nabla u\|_{L^2(B_1)},$$

where $\chi_{B_{\frac12}}$ is the characteristic function of the ball $B_{\frac12}$. Together with injections proved by Adams in [1], see also [5, 6.1.6], the latter shows that

$$\|\nabla^2 u\|_{L^r(B_{\frac13})}+\|\nabla u\|_{L^{2r}(B_{\frac13})}\leq C\big(\|f\|_p+\|\nabla^2 u\|_2+\|\nabla u\|_4\big),$$

for some $r>1$. Then bootstrapping this estimate, we get

$$\|\nabla^2 u\|_{L^{\bar p}(B_{\frac14})}+\|\nabla u\|_{L^{2\bar p}(B_{\frac14})}\leq C\big(\|f\|_p+\|\nabla^2 u\|_2+\|\nabla u\|_4\big),$$

where $\bar p$ is the limiting exponent of the bootstrapping given by the Sobolev injection of $W^{2,p}$ into $L^{\bar p}$ if $p<2$. Indeed, thanks to (2.7), the only limiting term for the bootstrap is the regularity of f.

Now, we can easily derive the proof of the energy gap. Indeed, thanks to the previous estimate, we easily see that for some $q>2$ we get

$$\|\nabla^2 u\|_{L^q(B_R)}+\|\nabla u\|_{L^{2q}(B_R)}\leq C\frac{\|u\|_{W^{2,2}}}{R^{2-\frac4q}}\quad\text{for all } R>0,$$

which proves that $u\equiv 0$. □

4 Uniform estimate in annular region

In this section, we derive a strong estimate for angular derivatives in an annular region independently of the conformal class.

Theorem 4.1. *There exist constants* $\varepsilon > 0$ *and* $C > 0$ *depending only on* k *such that if* $0 < r < \frac{1}{4}$, $p > 1$ *and* $u \in W^{2,2}(B_1 \setminus B_r, \mathbb{R}^k)$ *satisfies*

$$\Delta^2 u = \Delta(V\nabla u) + \operatorname{div}(w\nabla u) + \nabla\omega\nabla u + F\nabla u + f,$$

where

$$V \in W^{1,2}(B_1 \setminus B_r, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), \quad w \in L^2(B_1 \setminus B_r, \mathcal{M}_k),$$
$$\omega \in L^2(B_1 \setminus B_r, \mathrm{so}_k), \quad F \in L^2 \cdot W^{1,2}(B_1 \setminus B_r, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4),$$
$$f \in L^p(B_1, \mathbb{R}^k)$$

with

$$\|V\|_{W^{1,2}} + \|w\|_2 + \|\omega\|_2 + \|F\|_{L^2\cdot W^{1,2}} < \varepsilon,$$

then

$$\left\|\nabla^T\nabla u\right\|_{L^{2,1}(B_{\frac{1}{4}}\setminus B_{4r})} \le C\big(1 + \|\nabla^2 u\|_{L^2(B_1\setminus B_r)} + \|\nabla u\|_{L^4(B_1\setminus B_r)} + \|f\|_{L^p(B_1\setminus B_r)}\big),$$

where $\nabla^T f = \nabla f - \frac{\partial f}{\partial r}\frac{\partial}{\partial r}$.

Proof of Theorem 4.1. Using some Whitney extension theorem, we see that there exist

$$\tilde{V} \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), \quad \tilde{w} \in L^2(B_1, \mathcal{M}_k),$$
$$\tilde{\omega} \in L^2(B_1, \mathrm{so}_k), \quad \tilde{F} \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$$

such that $\tilde{V} = V$, $\tilde{w} = w$, $\tilde{\omega} = \omega$ and $\tilde{F} = F$ on $B_1 \setminus B_r$ and

$$\|\tilde{V}\|_{W^{1,2}} + \|\tilde{w}\|_2 + \|\tilde{\omega}\|_2 + \|\tilde{F}\|_{L^2\cdot W^{1,2}} < 2\varepsilon.$$

Thanks to Theorem 3.1, for $0 < \varepsilon < \frac{1}{2}$ small enough, there exist

$$A \in L^\infty \cap W^{2,2}(B_1, \mathcal{G}l_k) \quad \text{and} \quad B \in W^{1,(\frac{4}{3},1)}(B_1)$$

such that

$$d(A, \mathcal{SO}_k) + \|A\|_{W^{2,2}} + \|B\|_{W^{1,(\frac{4}{3},1)}} \le C\big(\|\tilde{V}\|_{W^{1,2}} + \|\tilde{w}\|_2 + \|\tilde{\omega}\|_2 + \|\tilde{F}\|_{L^2\cdot W^{1,2}}\big)$$

and

$$\nabla \Delta A + \Delta A V - \nabla A w + A(\nabla \omega + F) = \operatorname{curl} B.$$

Then we extend u by $\tilde{u} \in W^{2,2}(B_1)$ such that

$$\|\nabla^2 \tilde{u}\|_{L^2(B_1)} + \|\nabla \tilde{u}\|_{L^4(B_1)} \leq 2\big(\|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)}\big).$$

We easily see that $\tilde{u}$ satisfies

$$\Delta(A \Delta \tilde{u}) = \operatorname{div}(K) + Af \quad \text{on } B_1 \setminus B_r,$$

with

$$K = 2\nabla A \Delta \tilde{u} - \Delta A \nabla \tilde{u} + Aw \nabla \tilde{u} + \nabla A(V \nabla \tilde{u}) - A \nabla (V \nabla \tilde{u}) - B \nabla \tilde{u} \in L^{\frac{4}{3},1}(B_1)$$

such that

$$\|K\|_{L^{\frac{4}{3}}} \leq C\big(1 + \|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)}\big).$$

Then, we extend Af by $\tilde{f} \in L^p(B_1)$ such that

$$\|\tilde{f}\|_p \leq 2\|Af\|_p.$$

Then take $D \in W_0^{1,\frac{4}{3}}(B_1)$ which satisfies

$$\Delta D = \operatorname{div}(K) + \tilde{f} \quad \text{on } B_1.$$

Hence, thanks to the standard L^p-theory, there exists C a positive constant independent of r such that

$$\|D\|_{2,1} \leq C\big(\|K\|_{L^{\frac{4}{3},1}} + \|f\|_p\big).$$

Finally, thanks to Lemma 6.1, there exist $a, b \in \mathbb{R}^k$ and C a positive constant independent of r such that

$$\begin{aligned} &\left\| D - A\Delta u - a - \frac{b}{|x|^2} \right\|_{L^{2,1}(B_{\frac{1}{2}} \setminus B_{2r})} \\ &\quad \leq C \left\| D - A\Delta u \right\|_2 \\ &\quad \leq C\big(1 + \|\nabla^2 \tilde{u}\|_2 + \|K\|_{L^{\frac{4}{3},1}} + \|f\|_p\big). \end{aligned} \tag{4.1}$$

Hence we have

$$\operatorname{div}(A \nabla \tilde{u}) = a + \frac{b}{|x|^2} + F \quad \text{on } B_1 \setminus B_r$$

with

$$\|F\|_{L^{2,1}(B_{\frac{1}{2}} \setminus B_{2r})} \leq C\big(1 + \|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)} + \|f\|_p\big).$$

Let us proceed to the following Hodge decomposition, see [8, Corollary 10.5.1],

$$A d\tilde{u} = d\alpha + d^*\beta, \tag{4.2}$$

where $\alpha \in W_0^{1,2}(B_{\frac{1}{2}})$ and $\beta \in W^{1,2}(B_{\frac{1}{2}})$ satisfy

$$\Delta\alpha = a + \frac{b}{|x|^2} + F \quad \text{on } B_{\frac{1}{2}} \setminus B_{2r}$$

and

$$\Delta\beta = dA \wedge d\tilde{u} \quad \text{on } B_{\frac{1}{2}}.$$

On the one hand, we extend F by $\tilde{F} \in W^{1,2}(B_{\frac{1}{2}})$ such that

$$\|\tilde{F}\|_{L^{2,1}(B_{\frac{1}{2}})} \leq 2\|F\|_{L^{2,1}}.$$

Then, let $\tilde{\alpha} \in W_0^{1,2}(B_{\frac{1}{2}})$ which satisfies

$$\Delta\tilde{\alpha} = \tilde{F} \quad \text{on } B_{\frac{1}{2}}.$$

Hence, thanks to the standard bounds for singular integrals on Lorentz spaces, see [5], there exists C a positive constant independent of r such that

$$\|\nabla^2\tilde{\alpha}\|_{2,1} \leq C\|F\|_{2,1}.$$

Then, thanks to Lemma 6.1, there exists C a positive constant independent of r such that

$$\begin{aligned} &\left\|\nabla^T\nabla(\alpha - \tilde{\alpha})\right\|_{L^{2,1}(B_{\frac{1}{4}} \setminus B_{4r})} \\ &\quad \leq C\|\nabla^2(\alpha - \tilde{\alpha})\|_2 \\ &\quad \leq C\big(\|F\|_{2,1} + \|\nabla^2\beta\|_2 + \|\nabla A \nabla\tilde{u}\|_2 + \|A\nabla^2\tilde{u}\|_2\big). \end{aligned} \tag{4.3}$$

On the other hand, thanks to the standard-L^p-theory and Sobolev embeddings, we get

$$\left\|\nabla^2\beta\right\|_{L^{2,1}(B_{\frac{1}{4}})} \leq C\big(1 + \|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)}\big). \tag{4.4}$$

Here we use the injection of $W^{1,2}$ into $L^{4,2}$. Finally, thanks to (4.2), (4.3), (4.4) and the fact that

$$\left\|\nabla^T\nabla u\right\|_{L^{2,1}} \leq C\big(\left\|\nabla^T(A\nabla u)\right\|_{L^{2,1}} + \left\|\nabla^T A \nabla u\right\|_{L^{2,1}}\big),$$

we get the desired estimate. □

5 Proof of Theorem 1.2

First we are going to separate B_1 in three parts: one where u_n converges to a limiting solution, another composed of some small neighborhoods where the energy concentrates and where some bubbles blow and a third part which consists of some neck regions which join the first two parts. This “bubble-tree” decomposition is by now classical, see [13] for instance, hence we just sketch briefly how to proceed.

Step 1: Finding the points of concentration. Let ε_0 be such that the V, w, ω and F given by Section 2 satisfy, thanks to (2.7), the hypothesis of Theorem 3.3 as soon as $\|\nabla^2 u\|_2^2 + \|\nabla u\|_4^4 \le \varepsilon_0$. Then, thanks to (1.1), we easily proved that there exist finitely many points $a^1, \dots, a^n$ where

$$\int_{B(a_i,r)} \left(|\nabla^2 u|^2 + |\nabla u|^4\right) dx \ge \varepsilon_0 \quad \text{for all } r > 0. \tag{5.1}$$

Moreover, using Theorem 3.3, we prove that there exist $f \in L^p(B_1, \mathbb{R}^k)$ and an f-approximate biharmonic map $u_\infty \in W^{2,2}(B_1, N)$ such that, up to a subsequence,

$$f_n \rightharpoonup f \quad \text{in } L^p(B_1, \mathbb{R}^k)$$

and

$$\nabla u_n \to \nabla u_\infty \quad \text{in } W^{1,\bar{p}}_{\text{loc}}(B_1 \setminus \{a^1, \dots, a^n\}).$$

Step 2: Blow-up around a^i. We choose $r_i > 0$ such that

$$\int_{B(a_i,r_i)} \left(|\nabla^2 u_\infty|^2 + |\nabla u_\infty|^4\right) dx \le \frac{\varepsilon_0}{4}.$$

Then, we define a center of mass of $B(a^i, r^i)$ with respect to u_n in the following way:

$$a_n^i = \left(\frac{\int_{B(a^i,r^i)} x^\alpha |\nabla^2 u_n|^2 \, dx}{\int_{B(a^i,r^i)} |\nabla u_n|^2 \, dx} \right)_{\alpha=1,\dots,4}.$$

Let λ_n^i be a positive real such that

$$\int_{B(a_n^i,r^i)\setminus B(a_n^i,\lambda_n^i)} \left(|\nabla^2 u_n|^2 + |\nabla u_n|^4\right) dx = \frac{\varepsilon_0}{2}.$$

Then we set $\tilde{u}_n^i(x) = u_n(a_n^i + \lambda_n^i x)$ and $N_n^i = B(a_n^i, r^i) \setminus B(a_n^i, \lambda_n^i)$. Thanks to the conformal invariance, we easily see that

$$\int_{B(0,\frac{r^i}{\lambda_n^i})} \left(|\nabla^2 \tilde{u}_n^i|^2 + |\nabla \tilde{u}_n^i|^4\right) dx = \int_{B(a_n^i,r^i)} \left(|\nabla^2 u_n|^2 + |\nabla u_n|^4\right) dx \le M$$

and $\tilde{u}_n^i$ still satisfies the equation of approximate biharmonic maps with the approximation $(\lambda_n^i)^4 \tilde{f}_n$ which goes to zero in L^p-norm. Let a_i^j be the possible points of concentration of $\tilde{u}_n^i$ where

$$\int_{B(a_i^j,r)} \left(|\nabla^2 \tilde{u}_n^i|^2 + |\nabla \tilde{u}_n^i|^4\right) dz \geq \varepsilon_0 \quad \text{for all } r > 0. \tag{5.2}$$

Then, up of a subsequence, for each i,

$$\nabla \tilde{u}_k^i \to \nabla u_\infty^i \quad \text{in } W^{1,\bar{p}}_{\text{loc}}(B_1 \setminus \{a_i^1, \ldots, a_i^{n_i}\}),$$

where $u_\infty^i \in W^{2,2}(\mathbb{R}^4, N)$ is a biharmonic map.

Step 3: Iteration. Two cases have to be considered separately:

- $\tilde{u}_n^i$ is subject to some concentration phenomenon as (5.1), and then we find some new points of concentration, in such a case we apply Step 2 to our new concentration points.
- $\widetilde{u}_n^i$ converges in $W^{2,\bar{p}}_{\text{loc}}(\mathbb{R}^4)$ to a non-trivial biharmonic map.

Of course this process has to stop, since we are assuming a uniform bound on $\|\nabla^2 u_n\|_2 + \|\nabla^2 u_n\|_4$ and each step is consuming at least the energy of a non-trivial biharmonic map which is bounded from below thanks to the energy gap proved in Theorem 3.3.

Analysis of a neck region: A neck region is an annular region which is a union of a finite number of annuli $N_n^i = B(a_n^i, \mu_n^i) \setminus B(a_n^i, \lambda_n^i)$ such that

$$\lim_{k\to+\infty} \mu_n^i = 0, \quad \lim_{k\to+\infty} \frac{\lambda_n^i}{\mu_n^i} = 0,$$

and

$$\int_{N_n^i} \left(|\nabla^2 u_n|^2 + |\nabla u_n|^4\right) dx \leq \frac{\varepsilon_0}{2} \tag{5.3}$$

In order to prove Theorem 1.2, we start by proving a weak estimate on the energy of the gradient and the hessian in the region N_n^i.

First we remark that, for all $\varepsilon > 0$, there exists $r > 0$ such that for all $\rho > 0$ such that

$$B_{2\rho}(a_n^i) \setminus B_\rho(a_n^i) \subset N_n^i(r)$$

where $N_n^i(r) = B(a_n^i, r\mu_n^i) \setminus B(a_n^i, \frac{\lambda_n^i}{r})$, we have

$$\int_{B_{2\rho}(a_n^i)\setminus B_\rho(a_n^i)} \left(|\nabla^2 u_n|^2 + |\nabla u_n|^4\right) dx \leq \varepsilon. \tag{5.4}$$

If this is not the case there would exist a sequence $\rho_n^i \to 0$ such that, up to a subsequence,

$$\hat{u}_n = u_n(a_n^i + \rho_n^i z)$$

converges in $W^{2,\bar{p}}_{\mathrm{loc}}(\mathbb{R}^4 \setminus \{0\})$ to $\hat{u}_\infty$, a non-trivial biharmonic map. Using the fact that the $W^{2,2}$-norm of $\hat{u}_\infty$ is bounded and the Schwartz Lemma, we can remove the point singularity. Hence it has to be in fact a solution on the whole space. Using the energy gap proved in Theorem 3.3 we deduce that $\hat{u}_\infty$ is such that

$$\int_{N_k^i} \left(|\nabla^2 u_\infty|^2 + |\nabla u_\infty|^4\right) dx \geq \varepsilon_0, \tag{5.5}$$

which contradicts (5.3).

Then for all $\varepsilon > 0$, there exists $r > 0$ such that

$$\|\nabla^2 u_n\|_{L^{2,\infty}(N_n^i(r))} + \|\nabla u_n\|_{L^{4,\infty}(N_n^i(r))} \leq \varepsilon. \tag{5.6}$$

Indeed, let $0 < \varepsilon < \varepsilon_0$ and $r > 0$ such that, for all $B_{2\rho}(a_n^i) \setminus B_\rho(a_n^i) \subset N_n^i(r)$, we have

$$\int_{B_{2\rho}(a_n^i)\setminus B_\rho(a_n^i)} \left(|\nabla^2 u_n|^2 + |\nabla u_n|^4\right) dx \leq \varepsilon. \tag{5.7}$$

Then, thanks to ε-regularity in Theorem 3.3, there exist $q > 2$ and C a positive constant, independent of r and u, such that for all $\rho > 0$ such that

$$B_{2\rho}(a_n^i) \setminus B_\rho(a_n^i) \subset N_n^i\left(\frac{r}{2}\right),$$

and n big enough, we have

$$\begin{aligned} &\rho^{2-\frac{4}{q}} \|\nabla^2 u\|_{L^q(B_{2\rho}\setminus B_\rho)} + \rho^{1-\frac{2}{q}} \|\nabla u\|_{L^{2q}(B_{2\rho}\setminus B_\rho)} \\ &\quad \leq C\left(\sqrt{\varepsilon} + (r\mu_i^n)^{\frac{4(p-1)}{p}} |f_n|^p\right) \leq C\sqrt{\varepsilon}. \end{aligned} \tag{5.8}$$

Let $\lambda > 0$, $f(x) = |\nabla^2 u(x)|$ if $x \in N_n^i(\frac{r}{2})$ and $f = 0$ otherwise. For any $\rho > 0$, we denote

$$U(\lambda,\rho) \equiv \{x \in B_{2\rho} \setminus B_\rho \mid f(x) > \lambda\}.$$

Thanks to (5.8), we have

$$\lambda^q |U(\lambda,\rho)| \leq C^r \varepsilon^{\frac{q}{2}} \rho^{4-2q}.$$

Let $k \in \mathbb{Z}$ and $j \geq k$, we apply the previous inequality with $\rho = 2^{-j}\lambda^{-1}$ and we sum for $j \geq k$, which gives

$$\lambda^2 |\{x \in \mathbb{R}^4 \setminus B_{2^k\lambda^{-1}} \mid f(x) > \lambda\}| \leq C 2^{-k(4-2q)} \varepsilon^{\frac{r}{2}} \rho^{4-2q}.$$

Hence, for any $k \in \mathbb{Z}$, we have

$$\lambda^2 |\{x \in \mathbb{R}^4 \mid f(x) > \lambda\}| \leq C\big(2^{-k(4-2q)}\varepsilon^{\frac{q}{2}} + 2^{4k}\big).$$

Taking $2^{4k} \sim \varepsilon^{\frac{q}{2}}$, we have

$$\|\nabla^2 u_n\|_{L^{2,\infty}(N_n^i(r))} \leq C\varepsilon^{\frac{q}{4}},$$

We prove a similar inequality for $\|\nabla u_n\|_{L^{4,\infty}}$, and then we have (5.6).

Finally using Theorem 4.1 and the duality for Lorentz spaces, we see that, for all $\varepsilon > 0$, there exists $r > 0$ such that

$$\|\nabla^T(\nabla u)\|_{L^2(N_k^i(r))} \leq \varepsilon. \tag{5.9}$$

Then using the Pohožaev identity (7.4) for extrinsic biharmonic maps (resp. (7.5) for intrinsic biharmonic maps) and the fact that the convergence is strong on the boundary of a neck region, we get that for all $\varepsilon > 0$, there exists $r > 0$ such that

$$\|\nabla^2 u\|_{L^2(N_k^i(r))} + \|\nabla u\|_{L^4(N_k^i(r))} \leq \varepsilon. \tag{5.10}$$

Which achieves the proof of Theorem 1.2. □

Following step by step the proof of Theorem 1.2, we can prove the following theorem about the angular energy quantization of solution of fourth order elliptic system in the form of Lamm–Rivière, [10].

Theorem 5.1. *Let*

$$V_n \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), \quad w_n \in L^2(B_1, \mathcal{M}_k),$$
$$\omega_n \in L^2(B_1, \mathrm{so}_k), \qquad F_n \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4),$$

and let $u_n \in W^{2,1}(B_1, \mathbb{R}^n)$ be a sequence of solutions of

$$\Delta^2 u_n = \Delta(V_n \nabla u_n) + \mathrm{div}(w_n \nabla u_n) + \nabla\omega_n \nabla u_n + F_n \nabla u_n, \tag{5.11}$$

with bounded energy, i.e.

$$\|\nabla^2 u_n\|_2 + \|\nabla u_n\|_4 + \|V_n\|_{W^{1,2}} + \|w_n\|_2 + \|\omega_n\|_2 + \|F_n\|_{L^2.W^{1,2}} \leq M. \tag{5.12}$$

Then there exist

$$V_\infty \in W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), \quad w_\infty \in L^2(B_1, \mathcal{M}_k),$$
$$\omega_\infty \in L^2(B_1, \mathrm{so}_k), \qquad F_\infty \in L^2 \cdot W^{1,2}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4)$$

and let $u_\infty \in W^{2,1}(B_1, \mathbb{R}^n)$ be a solution of

$$\Delta^2 u_\infty = \Delta(V_\infty \nabla u_\infty) + \mathrm{div}(w_\infty \nabla u_\infty) + \nabla\omega_\infty \nabla u_\infty + F_\infty \nabla u_\infty \quad \text{on } B_1,$$

$l \in \mathbb{N}^*$ and

(i) $\theta^1, \dots, \theta^l$ *a family of solutions to a system of the form*

$$\Delta^2\theta^i = \Delta(V^i_\infty \nabla\theta^i) + \operatorname{div}(w^i_\infty \theta^i) + \nabla\omega^i_\infty \nabla\theta^i + F^i_\infty \nabla\theta^i \text{ on } \mathbb{R}^4$$

where

$$V^i_\infty \in W^{1,2}(\mathbb{R}^4, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4), \quad w^i_\infty \in L^2(\mathbb{R}^4, \mathcal{M}_k),$$
$$\omega^i_\infty \in L^2(\mathbb{R}^4, \mathrm{so}_k), \quad F^i_\infty \in L^2 \cdot W^{1,2}(\mathbb{R}^4, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4),$$

(ii) $a^1_n, \dots, a^l_n$ *a family of converging sequences of points of B_1,*

(iii) $\lambda^1_n, \dots, \lambda^l_n$ *a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$\begin{aligned}
V_n &\rightharpoonup V_\infty && \text{in } W^{1,2}_{\mathrm{loc}}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4),\\
w_n &\rightharpoonup w_\infty && \text{in } L^2_{\mathrm{loc}}(B_1, \mathcal{M}_k),\\
\omega_n &\rightharpoonup \omega_\infty && \text{in } L^2_{\mathrm{loc}}(B_1, \mathrm{so}_k),\\
F_n &\rightharpoonup F_\infty && \text{in } L^2_{\mathrm{loc}} \cdot W^{1,2}_{\mathrm{loc}}(B_1, \mathcal{M}_k \otimes \Lambda^1\mathbb{R}^4),\\
u_n &\to u_\infty && \text{on } W^{2,2}_{\mathrm{loc}}(B_1 \setminus \{a^1_\infty, \dots, a^l_\infty\})
\end{aligned}$$

and

$$\left\| \left\langle \nabla\left(\nabla\left(u_n - u_\infty - \sum_{i=1}^{l} \theta^i_k\right)\right), X_n \right\rangle \right\|_{L^2_{\mathrm{loc}}(B_1)} + \left\| \left\langle \nabla\left(u_n - u_\infty - \sum_{i=1}^{l} \theta^i_k\right), X_n \right\rangle \right\|_{L^4_{\mathrm{loc}}(B_1)} \to 0,$$

where $\omega^i_n = \omega^i(a^i_n + \lambda^i_n\,.)$ and X_n is any vector field whose image is in $(\nabla d_n)^\perp$ with $d_n = \min_{1\le i\le l}(\lambda^i_n + d(a^i_n, .))$.

6 A lemma about harmonic maps on an annular regions

Lemma 6.1. *Let $0 < r < \frac{1}{8}$ and $u \in W^{1,2}(B_1 \setminus B_r)$ be a harmonic function such that*

$$\int_{\partial B_1} u\, d\sigma = 0, \quad \int_{\partial B_r} u\, d\sigma = 0.$$

Then there exists C a positive constant independent of r and u such that

$$\|u\|_{L^{2,1}(B_{\frac{1}{2}} \setminus B_{2r})} \le C\|u\|_2 \quad \text{and} \quad \|\nabla^T \nabla u\|_{L^{2,1}(B_{\frac{1}{2}} \setminus B_{2r})} \le C\|\nabla^T \nabla u\|_2.$$

Proof. Since u is harmonic, it can be decomposed with respect to the spherical harmonics as follows:

$$u = \sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} \big(d_k^l r^l + d_k^{-l} r^{-l-2}\big)\phi_k^l, \tag{6.1}$$

where $(\phi_k^l)_{l,k}$ are a L^2-basis of eigenfunction of the Laplacian on S^3. In particular we get $\Delta\phi_k^l = -l(l+2)\phi_k^l$ on S^3. Thanks to this equation, L^p-theory for singular operators gives the existence of a positive constant C, independent of l such that $\|\phi_k^l\|_\infty \leq C(l(l+2))^2$.

Moreover we know that N_l, the dimension of the eigenspace associated to $-l(l+2)$, is equal to $(l+1)^2$. Hence, computing the L^2-norm and $L^{2,1}$-norm of the function $f_j : x \mapsto |x|^j$, we get

$$\begin{aligned}
\|f_j\|_2 &\geq \frac{r^{2+j}}{2\sqrt{-2j-4}} && \text{if } j < -2,\\
\|f_j\|_2 &\geq \frac{1}{2\sqrt{2j+4}} && \text{if } j \geq 0,\\
\|f_j\|_{L^{2,1}(B_{\frac12}\setminus B_{2r})} &\leq (2r)^{2+j} && \text{if } j < -2,\\
\|f_j\|_{L^{2,1}(B_{\frac12}\setminus B_{2r})} &\leq \left(\frac{1}{2}\right)^{\frac{3j}{4}+1} && \text{if } j \geq 0,
\end{aligned}$$

where C is independent of j.

Then

$$\begin{aligned}
\|u\|_{L^{2,1}(B_{\frac12}\setminus B_r)} &\leq C \sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} \left(d_k^l \left(\frac{1}{2}\right)^{\frac{3l}{4}+1} + d_k^{-l}(2r)^{-l}\right)(l(l+2))^2\\
&\leq C\left(\left(\sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} (d_k^l)^2 \frac{1}{4(2l+4)}\right)^{\frac12}\right.\\
&\qquad\qquad \times \left(\sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} 4(2l+4)(l(l+2))^4 \left(\frac{1}{2}\right)^{\frac{3l}{2}+2}\right)^{\frac12}\\
&\qquad + \left(\sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} (d_k^{-l})^2 \frac{r^{-2l}}{8l}\right)^{\frac12}\\
&\qquad\qquad \left. \times \left(\sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} 8l(l(l+2))^4 \left(\frac{1}{4}\right)^{l}\right)^{\frac12}\right).
\end{aligned}$$

Thanks to the fact that N_l, the dimension of the eigenspace associated to the eigenvalue $-l(l+2)$ of the Laplacian, is equal to $(l+1)^2$, we get the first estimate. The second identity is obtained in the same way. □

7 Pohožaev identities

In this section, we prove a Pohožaev identity for extrinsic and intrinsic biharmonic maps in order to rely the radial derivatives to the angular ones. First we multiply our equation by $x^k\partial_k u$ and we integrate by parts:

$$\begin{aligned}
&\int_{B(0,r)} (x^k\partial_k u)(\Delta^2 u)\,dx \\
&\quad = -\int_{B(0,r)} \langle \nabla u, \nabla(\Delta u)\rangle\,dx - \int_{B(0,r)} (x^k\partial_k\partial^i u)(\partial_i(\Delta u))\,dx \\
&\qquad + \int_{\partial B(0,r)} (x^k\partial_k u)\partial_\nu(\Delta u)\,d\sigma \\
&\quad = 2\int_{B(0,r)} (\Delta u)^2\,dx + \int_{B(0,r)} x^k\partial_k(\Delta u)(\Delta u)\,dx \\
&\qquad + \int_{\partial B(0,r)} \big((r\partial_\nu u)\partial_\nu(\Delta u) - (\partial_\nu u)(\Delta u) - r(\partial_\nu^2 u)(\Delta u)\big)\,d\sigma \\
&\quad = \int_{\partial B(0,r)} \frac{r}{2}(\Delta u)^2\,d\sigma \\
&\qquad + \int_{\partial B(0,r)} \big((r\partial_\nu u)\partial_\nu(\Delta u) - (\partial_\nu u)(\Delta u) - r(\partial_\nu^2 u)(\Delta u)\big)\,d\sigma.
\end{aligned}$$

Using the fact that for an extrinsic harmonic maps we have $\Delta^2 u \perp T_u N$ almost everywhere, we get for all r that

$$\int_{\partial B(0,r)} \left(\frac{1}{2}(\Delta u)^2 - (\partial_\nu^2 u)\Delta u + (\partial_\nu u)\partial_\nu(\Delta u) - \frac{1}{r}(\partial_\nu u)(\Delta u)\right) d\sigma = 0. \tag{7.1}$$

But

$$\Delta u = \partial_\nu^2 u + \frac{3}{r}\partial_\nu u + \frac{1}{r^2}\Delta_{S^3} u.$$

Hence

$$\begin{aligned}
(\Delta u)^2 &= (\partial_\nu^2 u)^2 + \frac{9}{r^2}(\partial_\nu u)^2 + \frac{1}{r^4}(\Delta_{S^3} u)^2 + \frac{6}{r}(\partial_\nu u)(\partial_\nu^2 u) \\
&\quad + \frac{2}{r^2}(\Delta_{S^3} u)(\partial_\nu^2 u) + \frac{6}{r^3}(\partial_\nu u)(\Delta_{S^3} u).
\end{aligned}$$

On the one hand, we have

$$\frac{1}{2}(\Delta u)^2-(\partial_\nu^2 u)\Delta u = -\frac{1}{2}(\partial_\nu^2 u)^2+\frac{9}{2r^2}(\partial_\nu u)^2+\frac{1}{2r^4}(\Delta_{S^3} u)^2+\frac{3}{r^3}(\partial_\nu u)(\Delta_{S^3} u),$$

which gives

$$\begin{aligned}\int_{B_R\setminus B_r}&\left(\frac{1}{2}(\Delta u)^2-(\partial_\nu^2 u)\Delta u\right)dx\\ &=\int_{B_R\setminus B_r}\left(-\frac{1}{2}(\partial_\nu^2 u)^2+\frac{9}{2r^2}(\partial_\nu u)^2\right.\\ &\qquad\left.+\frac{1}{2r^4}(\Delta_{S^3} u)^2+\frac{3}{r^3}(\partial_\nu u)\,(\Delta_{S^3} u)\right)dx.\end{aligned}\tag{7.2}$$

On the other hand

$$\begin{aligned}(\partial_\nu u)\partial_\nu(\Delta u)-\frac{1}{r}(\partial_\nu u)(\Delta u) &= (\partial_\nu u)(\partial_\nu^3 u)+\frac{2}{r}(\partial_\nu u)(\partial_\nu^2 u)-\frac{6}{r}(\partial_\nu u)^2\\ &\quad+\frac{1}{r^2}(\partial_\nu \Delta_{S^3} u)(\partial_\nu u)-\frac{3}{r^3}(\Delta_{S^3} u)(\partial_\nu u).\end{aligned}$$

Integrating by part, we get

$$\begin{aligned}\int_{B_R\setminus B_r}&\left((\partial_\nu u)\partial_\nu(\Delta u)-\frac{1}{r}(\partial_\nu u)(\Delta u)\right)dx\\ &=\int_{B_R\setminus B_r}\left((\partial_\nu u)(\partial_\nu^3 u)+\frac{2}{r}(\partial_\nu u)(\partial_\nu^2 u)-\frac{6}{r}(\partial_\nu u)^2\right)dx\\ &\qquad+\int_{B_R\setminus B_r}\left(\frac{1}{r^2}(\partial_\nu \Delta_{S^3} u)(\partial_\nu u)-\frac{3}{r^3}(\Delta_{S^3} u)(\partial_\nu u)\right)dx\\ &=\int_{\partial(B_R\setminus B_r)}(\partial_\nu u)(\partial_\nu^2 u)\,d\sigma\\ &\qquad+\int_{B_R\setminus B_r}\left(-\frac{1}{2r}(\partial_\nu(\partial_\nu u)^2)-(\partial_\nu^2 u)^2-\frac{6}{r}(\partial_\nu u)\right)dx\\ &\qquad+\int_{B_R\setminus B_r}\left(\frac{1}{r^2}(\partial_\nu \Delta_{S^3} u)(\partial_\nu u)-\frac{3}{r^3}(\Delta_{S^3} u)(\partial_\nu u)\right)dx\\ &=\int_{\partial(B_R\setminus B_r)}\left((\partial_\nu u)(\partial_\nu^2 u)-\frac{1}{2r}(\partial_\nu u)^2\right)d\sigma\\ &\qquad-\int_{B_R\setminus B_r}\left((\partial_\nu^2 u)^2+\frac{5}{r^2}(\partial_\nu u)^2\right)dx\\ &\qquad+\int_{B_R\setminus B_r}\left(\frac{1}{r^2}(\partial_\nu \Delta_{S^3} u)(\partial_\nu u)-\frac{3}{r^3}(\Delta_{S^3} u)(\partial_\nu u)\right)dx.\end{aligned}\tag{7.3}$$

Finally, thanks to (7.1), (7.2) and (7.3), we have

$$\begin{aligned}\int_{B_R\setminus B_r}&\left(\frac{3}{2}(\partial_\nu^2 u)^2+\frac{1}{2r^2}(\partial_\nu u)^2\right)dx\\&=\int_{B_R\setminus B_r}\left(\frac{1}{2r^4}(\Delta_{S^3}u)^2\right)dx+\int_{B_R\setminus B_r}\left(\frac{1}{r^2}(\partial_\nu\Delta_{S^3}u)(\partial_\nu u)\right)dx\\&\quad+\int_{\partial(B_R\setminus B_r)}\left((\partial_\nu u)(\partial_\nu^2 u)-\frac{1}{2r}(\partial_\nu u)^2\right)d\sigma.\end{aligned}\tag{7.4}$$

Since the equations of extrinsic and intrinsic biharmonic maps differ only by $P(u)(B(u)(\nabla u,\nabla u)\nabla_u B(u)(\nabla u,\nabla u))+2B(u)(\nabla u,\nabla u)B(u)(\nabla u,\nabla P(u))$, we multiply this term by $x^k\partial_k u$ which gives

$$\begin{aligned}x^k\partial_k u\Big(P(u)\big(&B(u)(\nabla u,\nabla u)\nabla_u B(u)(\nabla u,\nabla u)\big)\\&+2B(u)(\nabla u,\nabla u)B(u)(\nabla u,\nabla P(u))\Big)\\&=B(u)(\nabla u,\nabla u)\nabla_{x^k\partial_k u}B(u)(\nabla u,\nabla u)\\&\quad+2B(u)(\nabla u,\nabla u)B(u)(\nabla u,\nabla(x^k\partial_k u)\\&=x^k\partial_k\left(\frac{|B(u)(\nabla u,\nabla u)|^2}{2}\right)+2\,|B(u)(\nabla u,\nabla u)|^2\\&=\frac{1}{|x|^3}\frac{\partial}{\partial\nu}\left[\frac{r^4}{2}\,|B(u)(\nabla u,\nabla u)|^2\right].\end{aligned}$$

Then integrating, we get the following Pohoždev identity for intrinsic biharmonic maps:

$$\begin{aligned}\int_{B_R\setminus B_r}&\left(\frac{3}{2}(\partial_\nu^2 u)^2+\frac{1}{2r^2}(\partial_\nu u)^2\right)dx\\&=\int_{B_R\setminus B_r}\left(\frac{1}{2r^4}(\Delta_{S^3}u)^2\right)dx+\int_{B_R\setminus B_r}\left(\frac{1}{r^2}(\partial_\nu\Delta_{S^3}u)(\partial_\nu u)\right)dx\\&\quad+\int_{\partial(B_R\setminus B_r)}\left((\partial_\nu u)(\partial_\nu^2 u)-\frac{1}{2r}(\partial_\nu u)^2-\frac{r}{2}|B(u)(\nabla u,\nabla u)|^2\right)d\sigma.\end{aligned}\tag{7.5}$$

We also get a Pohoždev identity for the critical point of general functional, since

$$\begin{aligned}\int_{B_R\setminus B_r}&(x^k\partial_k u)H\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\frac{\partial u}{\partial x_3},\frac{\partial u}{\partial x_4}\right)dx\\&=\int_{B_R\setminus B_r}d\Omega\left(x^k\frac{\partial u}{\partial x_k},\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\frac{\partial u}{\partial x_3},\frac{\partial u}{\partial x_4}\right)dx=0.\end{aligned}$$

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